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A NOTE ON THE MONOMIAL CONJECTURE

S. P. DUTTA

Abstract. Several cases of the monomial conjecture are proved. An equivalent form of the direct summand conjecture is discussed.

Let (A, m, k) be a noetherian local ring of dimension n, m its maximal ideal and $k = A/m$. The Monomial Conjecture (henceforth MC) of Hochster asserts that, given any system of parameters (henceforth s.o.p.) $x_1, \ldots, x_n$ of A,

$$(x_1 x_2 \ldots x_n)^{t-1} \notin (x_1^t, \ldots, x_n^t) \quad \forall t > 0.$$

Hochster proved the conjecture in the equicharacteristic case [H1], [H2]. He also established the fact that in the mixed and the positive characteristic cases the Direct Summand Conjecture (henceforth DSC) and hence MC is equivalent to the Canonical Element Conjecture [H2] (henceforth CEC). Thus MC occupies a central position in the study of several homological conjectures. In [H1] Hochster pointed out that, given any s.o.p. $x_1, \ldots, x_n$ of A, $x_1^t, \ldots, x_n^t$ satisfies MC for t sufficiently large. Next Goto [G] proved MC for Buchsbaum rings and Koh proved DSC for degree p extensions [K]. Several special cases of CEC were proved in [D1] when $depth A = dim A - 1$. In [D2], the following result was established: If $J_i = Ann H_m^{n-i}(A)$ and $J = J_1 J_2 \ldots J_r$ where $r = dim A - depth A > 0$, then $x_1, \ldots, x_n$ satisfies MC if $J \not\subset (x_1, \ldots, x_n)$. This in turn implies that, given any s.o.p. $x_1, \ldots, x_n$ in a complete local normal domain A, $x_1, x_2, x_3^t, \ldots, x_n^t$ satisfies MC for $t \gg 0$. We will have several more applications of this result in Section 2. We recall that there is no loss of generality in assuming A to be a complete local normal domain. In our most recent work we established the validity of CEC over i) A/xA when A is a complete local normal domain and $x \in mJ_1$ [D3], ii) any almost complete intersection domain A or any almost complete intersection ring A over which p (= the mixed characteristic) is a non-zero-divisor [D4], iii) rings of the form R/Ω, where R is a complete Gorenstein ring such that the complete local normal domain A is a homomorphic image of R, $dim R = dim A$, and Ω is the canonical module of A [D4], and iv) complete local normal domains A for which Ω is S_3. Thus MC holds in all the above cases.

The main results of this paper are arranged in the following way.

In Section 1, first we reduce the study of MC to almost complete intersection rings. Recall that, due to the results mentioned earlier, the problem boils down to almost complete intersection rings of depth 0. Next we prove the following:

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Theorem (1.3). *MC holds over all local rings if and only if for every almost complete intersection ring A and for every s.o.p. $x_1,\ldots,x_n$ of A, $\ell(A/\underline{x}) > \ell(H_1(\underline{x};A))$ ($H_1(\underline{x};A)$ denotes H_1 of the Koszul complex $K_\bullet(\underline{x};A)$ on $x_1,\ldots,x_n$).*

As a corollary we derive:

Corollary. *Over an almost complete intersection ring A, $x_1,\ldots,x_n$ satisfies MC in the following cases*:
a) when $H_1(\underline{x};A)^v$ is not cyclic ($M^v = Hom_A(M,E(k))$);
and
b) when $H_1(\underline{x};A)$ is decomposable.

So, the crucial case is when $H_1(\underline{x};A)^v$ is cyclic.
Our next theorem states:

Theorem (1.6). *Given an s.o.p. $x_1,\ldots,x_n$ of a complete local domain A, we can construct $y_1,\ldots,y_{n-1}$, each $y_i \in (x_1,\ldots,x_n)$ for $i = 1,\ldots,n-1$, such that the ideal $(y_1,\ldots,y_{n-1},x_n)$ = the ideal $(x_1,\ldots,x_n)$ and $y_1,\ldots,y_{n-1},x_n^t$ satisfies MC for $t \gg 0$.*

Recall that $x_1,\ldots,x_n$ satisfies MC if and only if $y_1,\ldots,y_{n-1},x_n$, as above, satisfies the same. Our proof exploits the Hilbert-Samuel multiplicity to a large extent.
In Section 2, our first result states the following:

Corollary 1(2.1). *Any s.o.p. $x_1,\ldots,x_n$ of A with $x_n \in mJ$ satisfies MC.*

Corollary (2). *Given a local ring A, there exists a positive integer r such that for every s.o.p. $x_1,\ldots,x_n$ of A, $x_1^r,\ldots,x_n^r$ satisfies MC.*

We also establish the following:

Proposition (2.2). *Given an s.o.p. $x_1,\ldots,x_n$ of A, we can construct $y_1,\ldots,y_{n-1}$, such that $(y_1,\ldots,y_{n-1},x_j) = (x_1,\ldots,x_n)$ for some $j \in [1,\ldots,n]$, and for every $z \in J$ for which $y_1,\ldots,y_{n-1},z$ is an s.o.p., $y_1,\ldots,y_{n-1},x_jz$ satisfies MC.*

In Section 3, we raise the following question. Let R be a complete regular local ring and let $S = R[Y_1,\cdots,Y_d]/(F_1(Y_1),\cdots,F_d(Y_d))$, where $F_i(Y)$ is a monic irreducible polynomial in $R[Y]$. Then S is a free R-module.

Question. Does S possess a zero-divisor which is also a minimal generator of S over R?

In (3.1) we show how this question is related to DSC and in (3.2) we prove that the answer is in the affirmative in the following cases:
i) when R contains a field
and
ii) when S has a minimal prime P such that S/P is normal.

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1.1. Proposition. *Let R be a complete regular local ring and let $i : R \to A$ be a local module-finite extension. Let m denote the maximal ideal of R and let $y_1,\ldots,y_d$ generate the maximal ideal m_A in A. Let $F_i(Y)$ denote the monic polynomial of least degree in $R[Y]$ satisfied by y_i for $i = 1,2,\cdots,d$. Write $S = R[Y_1,\ldots,Y_d]/(F_1(Y_1),\ldots,F_d(Y_d))$ and let $\phi : S \to A$ be such that $\phi(Y_i) = y_i$. Then $Hom_R(A,R) \simeq Hom_S(A,S)$ and*

$$Hom_S(A,S) \subset mS \Longleftrightarrow Im\, Hom_R(A,R)\ (= i^*Hom_R(A,R)) \subset m.$$

Proof. Let $I = Ker\ \phi$. So $A \simeq S/I$. Note that, for every i, all the coefficients of $F_i(Y)$, except the leading one, are in m. Recall that

$$Hom_R(A,R) = \{f \in Hom_R(S,R) | f(I) = 0\}.$$

$\mathrm{Hom}_R(S,R)$ has a structure of an S-module: $f \in \mathrm{Hom}_R(S,R), \lambda \in S, (\lambda f)(x) = f(\lambda x)$. Since S is a module-finite extension of R, $\mathrm{Hom}_R(S,R)$ is a canonical module over S; and since S in Gorenstein, $\mathrm{Hom}_R(S,R)$ is isomorphic to S as S-modules.

Now

$$\begin{aligned} Hom_R(A,R) &= Hom_R(S/I,R) \\ &\simeq Hom_S(S/I,\ Hom_R(S,R)) \\ &\simeq Hom_S(A,S). \end{aligned}$$

And it is easy to check that $\mathrm{Hom}_S(A,S) \subset mS \Longleftrightarrow Im\ \mathrm{Hom}_R(A,R) \subset m$. □

Corollary. *i splits as an R-module map* $\Longleftrightarrow Hom_S(A,S) \not\subset mS$.

Remarks. 1. The proof of the above proposition and the corollary are valid even when R is a complete Gorenstein ring.

2. In the fall of 1993, I was visiting P. Roberts at the University of Utah. When I told him about the above result, he made me aware of the following result due to Strooker and Stückrad.

Main Result, [Str–Stü]**.** *Let S be any complete intersection. Then MC is equivalent to the truth of* (P) *for all complete intersection ring S where* (P) *is the following*:

(P): *Let α be an ideal in the ring S of height* 0. *Then $Ann_S\alpha$ is not contained in any parameter ideal of S.*

The above result of Strooker and Stückrad has already appeared in print—moreover it is more general than the proposition we had in the beginning of this section. But our proofs are completely different. Since their result can be used in a more straightforward manner, in this section, by (1.1) we will always mean their Main Result. We will get back to our proposition in Section 3.

Mostly we will use the following: Let $x_1,\ldots,x_n$ be an s.o.p. of A and let Ω be the canonical module $\mathrm{Hom}_S(A,S)$, where S is a complete intersection ring such that $A = S/I$ and $dimS = dimA$. If $x'_1,\ldots,x'_n$ denote a lift of $x_1,\ldots,x_n$ in S such that $x'_1,\ldots,x'_n$ is an s.o.p. of S, then $x_1,\ldots,x_n$ satisfies MC if and only if $\Omega \not\subset (x'_1,\ldots,x'_n)$. Henceforth we will also denote this lift $x'_1,\ldots,x'_n$ by $x_1,\ldots,x_n$, respectively, when there is no room for confusion.

1.2. Proposition. *MC holds for all local rings if and only if MC holds for all local almost complete intersections.*

Proof. In one direction there is nothing to prove! Let us assume that MC holds for all local almost complete intersections.

Let A be a complete local domain. Then $A = S/P$, where S is a complete intersection and $P \in Ass(S)$ is such that $PS_P = 0$. Write $\Omega = \mathrm{Hom}_S(A,S)$. Then we can find $\lambda \in P$ such that $\lambda \notin U\{\mathfrak{q}/\mathfrak{q} \in Ass(S), \mathfrak{q} \neq P\}$. It is easy to see that $\Omega = \mathrm{Hom}_S(S/\lambda S, S)$. By hypothesis MC is valid over $S/\lambda S$. So Ω is not contained in any parameter ideal of S, by (1.1). Hence A satisfies MC (1.1). □

1.3. Theorem. *MC is valid if and only for any local almost complete intersection A and for any s.o.p. $x_1,\ldots,x_n$ of A, $\ell(A/\underline{x}) > \ell(H_1(\underline{x};A))$. (Here $\underline{x}$ stands for the ideal $(x_1,\ldots,x_n)$ and $H_1(\underline{x};A)$ stands for H_1 of the Koszul complex $K_\bullet(x_1,\ldots,x_n;A)$).*

Proof. By the above proposition we need to prove MC only for local almost complete intersections. Let $A = S/\lambda S$, where S is a local complete intersection and $dimA = dimS$. Write $\Omega =$Hom$(S/\lambda S, S)$. We have the following short exact sequence:

$$0 \to S/\Omega \to S \to S/\lambda S \to 0,\ldots \qquad (1)$$
$$\overline{1} \to \lambda.$$

Suppose MC holds. Let $x_1,\ldots,x_n$ be any s.o.p. in A and let $x'_1,\ldots,x'_n$ be a lift of $x_1,\ldots,x_n$, respectively, in S such that $x'_1,\ldots,x'_n$ form an s.o.p. in S. Then $\Omega \not\subset (x'_1,\ldots,x'_n)$ (1.1). Applying $\otimes S/\underline{x}'S$ to (1), we get

$$0 \to H_1(\underline{x};A) \to S/\Omega + \underline{x}'S \to S/\underline{x}'S \to A/\underline{x}A \to 0\ldots. \qquad (2)$$

Hence $\ell(A/\underline{x}A) - \ell(H_1(\underline{x};A)) = \ell(S/\underline{x}'S) - \ell(S/\Omega + \underline{x}'S) > 0$ (as $\Omega \not\subset \underline{x}'S$). Conversely if $\ell(A/\underline{x}A) > \ell(H_1(\underline{x};A))$, it follows that $\ell(S/\underline{x}'S) > \ell(S/\Omega + \underline{x}'S) \Longrightarrow \Omega \not\subset (\underline{x}')$. □

Corollary. *Let A be a local almost complete intersection. Let $x_1,\ldots,x_n$ be an s.o.p. of A. Then $x_1,\ldots,x_n$ satisfies MC in the following cases:*
a) $H_1(\underline{x};A)^v$ (=Hom$(H_1(\underline{x};A),E(k)); E(k)$ = injective hull of k) is not cyclic.
b) $H_1(\underline{x};A)^v$ is decomposable.

The proof is immediate from the fact that if $\Omega \subset \underline{x}'S$, then

$$H_1(\underline{x};A) = \mathrm{Hom}(S/\lambda S, S/\underline{x}'S) = E_{A/\underline{x}A}(k).$$

Remarks. 1. It follows from the above that for successful completion of MC, the crucial case is the study of almost complete intersections A for which $H_1(\underline{x};A)^v$ is cyclic.

2. We would like to remind the reader that the proof of Theorem (2.2) in [D4] shows that, by induction on the dimension of A, the CEC (hence MC) is valid for almost complete intersection rings of positive depth.

1.4. Before proceeding to prove our next theorem (1.6), we recall certain facts about superficial elements in a local ring A. For the definition and properties of superficial elements we refer the reader to [N] and [Sa]. In [Sa], Samuel proved that, given any s.o.p. $x_1,\ldots,x_n$ of A, $\exists\, y_1,\ldots,y_{n-1} \in (x_1,\ldots,x_n)$ such that i) each y_i is a superficial element in $(x_1,\ldots,x_n)$, ii) $(y_1,\ldots,y_{n-1},x_n) = (x_1,\ldots,x_n)$, and iii) (Hilbert multiplicity) $e(y_1,\ldots,y_{n-1},x_n;A) = e(y_2,\ldots,y_{n-1},x_n;A/y_1A)$ $= e(y_3,\ldots,y_{n-1},x_n;A/(y_1,y_2)A) = \cdots = e(x_n,A/(y_1,\ldots,y_{n-1}))$.

Proposition. *Let $x_1,\ldots,x_n$ be an s.o.p. of a local ring A such that*

$$e(x_1,\ldots,x_n;A) = e(x_2,\ldots,x_n;A/x_1A) = \cdots = e(x_n;A/(x_1,\ldots,x_{n-1})).$$

Then $H_i(x_1,\ldots,x_{n-1};A)$ is a module of finite length for every $i > 0$.

Proof. We abbreviate the ideal $(x_1,\ldots,x_{n-1})$ by $\underline{x}_{n-1}$. Let $P_1,\ldots,P_r$ be the minimal primes of $\underline{x}_{n-1}$. Then

$$e(x_n;A/\underline{x}_{n-1}) = \sum_{i=1}^{r} \ell((A/\underline{x}_{n-1})_{P_i})e(x_n;A/P_i). \qquad (1)$$

On the other hand, by the associativity property for Hilbert multiplicity, we have

$$e(x_1, \dots, x_n; A) = \sum_{i=1}^{r} e(x_n; A/P_i) e(\underline{x}_{n-1}; A_{P_i}). \tag{2}$$

We refer to [N] for (1) and (2). Recall that

$$e(\underline{x}_{n-1}; A_{P_i}) = \sum_{j=0}^{n-1} (-1)^j H_j(\underline{x}_{n-1}; A_{P_i})$$

[Se]. Write $\chi_1^{A_{P_i}} = \sum_{j=0}^{n-2} (-1)^j H_{j+1}(\underline{x}_{n-1}; A_{P_i})$. Let us recall from [L] that $\chi_1^{A_{P_i}} \geq 0$ and $= 0$ if and only if $H_j(\underline{x}_{n-1}; A_{P_i}) = 0$ for $j \geq 1$. Now subtracting (2) from (1), we get

$$0 = \sum_{i=1}^{r} \chi_1^{A_{P_i}} \cdot e(x_n; A/P_i).$$

Since $e(x_n; A/P_i) > 0$, we must have $\chi_1^{A_{P_i}} = 0$ for every $i = 1, 2, \dots, r$.

$\Longrightarrow H_j(\underline{x}_{n-1}; A_{P_i}) = 0 \quad$ for every $\quad j \geq 1 \quad$ and for every $\quad i = 1, 2, \dots, r$

$\Longrightarrow \ell(H_j(\underline{x}_{n-1}; A)) < \infty, \quad$ for every $\quad j \geq 1. \quad$ (Recall : $dim H_j(\underline{x}_{n-1}; A) \leq 1$.) □

1.5. Proposition. *Let A be a local ring and let $x_1, \dots, x_n$ be an s.o.p. of A. Then there exist $y_1, \dots, y_{n-1}$ in $(x_1, \dots, x_n)$ such that $(y_1, \dots, y_{n-1}, x_n) = (x_1, \dots, x_n)$ and $\ell(A/(y_1, \dots, y_{n-1}, x_n^t)) > \ell(H_1(y_1, \dots, y_{n-1}, x_n^t; A))$ for $t \gg 0$.*

Proof. Write $\underline{y}_{n-1}$ for $(y_1, \dots, y_{n-1})$ and $\underline{x}$ for $(x_1, \dots, x_n)$. Following Samuel [Sa], we choose $y_1, \dots, y_{n-1}$ in $\underline{x}$ such that each y_i is a superficial element in $\underline{x}$ and $(\underline{y}_{n-1}, x_n)$ satisfy the properties i), ii) and iii) mentioned at the beginning of 1.4. Then $\ell(H_j(\underline{y}_{n-1}; A)) < \infty$ for every $j \geq 1$. We consider the following exact sequence:

$$0 \to \frac{H_1(\underline{y}_{n-1}; A)}{x_n H_1(\underline{y}_{n-1}; A)} \to H_1(\underline{y}_{n-1}, x_n; A) \to (0 : x_n)A/\underline{y}_{n-1} \to 0.$$

Note that for $t \gg 0$ we have: a) $x_n^t H_1(\underline{y}_{n-1}; A) = 0$, and b) $\ell((0 : x_n^t)A/\underline{y}_{n-1})$ is constant. Moreover, for every $t > 0$, $\ell(A/(\underline{y}_{n-1}, x_n^t)) - \ell((0 : x_n^t)A/\underline{y}_{n-1}) = e(x_n^t; A/\underline{y}_{n-1}) > 0$. As t increases, $\ell(A/(\underline{y}_{n-1}, x_n^t))$ increases, and hence there is a $t \gg 0$ for which

$$e(x_n^t; A/\underline{y}_{n-1}) > \ell(H_1(\underline{y}_{n-1}; A)) = \ell(H_1(\underline{y}_{n-1}; A)/x_n^t H_1(\underline{y}_{n-1}; A)).$$

Thus

$$\ell(A/(\underline{y}_{n-1}, x_n^t)) > \ell(H_1(\underline{y}_{n-1}, x_n^t; A)). \quad \square$$

1.6. Theorem. *Let A be a complete local domain and let $x_1, \dots, x_n$ be an s.o.p. of A. Then there exist $y_1, \dots, y_{n-1} \in (x_1, \dots, x_n)$ such that $(y_1, \dots, y_{n-1}, x_n) = (x_1, \dots, x_n)$ and $y_1, \dots, y_{n-1}, x_n^t$ satisfies MC for $t \gg 0$.*

Proof. As pointed out in the proof of Proposition (1.2), we can replace A by $S/\lambda S$, where S is a complete intersection, $A = S/P$, $dim S = dim A$ and λ is an element in P suitably chosen, such that $\Omega = \mathrm{Hom}_S(A, S) = \mathrm{Hom}_S(S/\lambda S, S)$. Moreover $x_1, \dots, x_n$ can be lifted to an s.o.p. in S, and hence can assume $x_1, \dots, x_n$ (actually images of $x_1, \dots, x_n$ respectively) is an s.o.p. in $S/\lambda S$. Write $B = S/\lambda S$.

By Proposition 1.1, it is enough to prove our theorem over B. By Theorem (1.3), $x_1, \ldots, x_n$ satisfies MC if and only if $\ell(B/\underline{x}B) > \ell(H_1(\underline{x}; B))$. By Proposition (1.5) we can choose $y_1, \ldots, y_{n-1} \in (x_1, \ldots, x_n)$ such that

$$(y_1, \ldots, y_{n-1}, x_n) = (x_1, \ldots, x_{n-1}, x_n)$$

and $\ell(B/(y_1, \ldots, y_{n-1}, x_n^t)) > \ell(H_1(y_1, \ldots, y_{n-1}, x_n^t; B))$ for $t \gg 0$. So by Theorem (1.3), $y_1, \ldots, y_{n-1}, x_n^t$ satisfies MC over B, and hence over A (Proposition (1.1)). □

1.7. *Remark.* We noted in (1.3) that for an s.o.p. $x_1, \ldots, x_n$, in an almost complete intersection A, to satisfy MC it is necessary and sufficient that $\ell(A/\underline{x}A) > \ell(H_1(\underline{x}; A))$ (here $\underline{x}$ stands for $(x_1, \ldots, x_n)$). In our proofs of Proposition (1.5) and Theorem (1.6) we understood that by modifying $x_1, \ldots, x_n$, if necessary, we can assume $e(\underline{x}; A) = e(x_2, \ldots, x_n; A/x_1A) = \cdots = e(x_n; A/\underline{x}_{n-1})$. Here $\underline{x}_{n-1}$ stands for $(x_1, \ldots, x_{n-1})$. Thus, in order that $x_1, \ldots, x_n$ satisfy MC, we must have $e(x_n; A/\underline{x}_{n-1}) > \ell(H_1(\underline{x}_{n-1}; A)/x_nH_1(\underline{x}_{n-1}; A))$. This implies, by a simple checking, that for $x_1, \ldots, x_n$ to satisfy MC on an almost complete intersection ring A, we must have

$$e(\underline{x}; A) > \chi_2^A(\underline{x}; A) = \sum_{i \geq 2}^{n} (-1)^i \ell(H_i(\underline{x}; A)).$$

2

In this section we are going to explore some ramifications of the following theorem.

2.1. Let A be a local ring and let J_i denote $Ann_A H_m^{n-i}(A)$. Let $J = J_1, \ldots, J_r$ where $r = dimA - depthA$.

Theorem. *Let $x_1, \ldots, x_n$ be an s.o.p. of A such that $J \not\subset (x_1, \ldots, x_n)$. Then $x_1, \ldots, x_n$ satisfies MC.*

For proof we refer the reader to Theorem 2.3 in [D2].

Corollary 1. *Let $x_1, \ldots, x_n$ be an s.o.p. of A such that $x_n \in mJ$. Then $x_1, \ldots, x_n$ satisfies MC.*

Proof. By hypothesis, $\ell(A/J + (x_1, \ldots, x_{n-1})) < \infty$. Hence $J \not\subset (x_1, \ldots, x_{n-1})$ and the image of J is an ideal of height 1 in $A/(x_1, \ldots x_{n-1})$. Since $x_n \in mJ$, J cannot be contained in $(x_1, \ldots, x_n)$. So we are done by the above theorem. □

Remark. We would like to remind the reader of an important result on CEC in this context: Theorem 3.1 of [D3] says that if x is a non-zero divisor in mJ_1, then A/xA satisfies CEC.

Corollary 2. *Let A be a local ring. Then there exists a positive integer r such that for every s.o.p. $x_1, \ldots, x_n$ of A, $x_1^r, \ldots, x_n^r$ satisfies MC.*

Proof. Since J is an ideal of positive height, $J \neq 0$, and hence $\exists r > 1$ such that $J \subset m^{r-1} - m^r$. Hence, for any s.o.p. $x_1, \ldots, x_n$ of A, $J \not\subset (x_1^r, \ldots, x_n^r)$. The above theorem now finishes off the proof. □

2.2. Proposition. *Given an s.o.p.* $x_1, \dots, x_n$ *of* A*, we can find* $y_1, \dots, y_{n-1}$ *such that* $(x_1, \dots, x_n) = (y_1, \dots, y_{n-1}, x_j)$ *for some* $j \in [1, 2, \dots, n]$ *and* $y_1, \dots, y_{n-1}, x_j z$ *satisfies MC for every* $z \in J$ *such that* $y_1, \dots, y_{n-1}, z$ *is an s.o.p. of* A *(*J *as in 2.1).*

Proof. Recall that $htJ \geq 1$. Write $\overline{A}$ for A/J, and $\overline{x_i}$ for the image of x_i in $\overline{A}$. Then $\ell(\overline{A}/(\overline{x}_1, \dots, \overline{x}_n)) < \infty$. Hence we can find elements $y_1, \dots, y_{n-1}$ in A such that i) $(\overline{y}_1, \dots, \overline{y}_{n-1}) \subset (\overline{x}_1, \dots, \overline{x}_n)$, ii) $\overline{y}_1, \dots, \overline{y}_{n-1}$ contains an s.o.p. of $\overline{A}$, and iii) $(y_1, \dots, y_{n-1}, x_j) = (x_1, \dots, x_n)$ for some $j \in [1, \dots, n]$. Let $z \in J$ be such that $y_1, \dots, y_{n-1}, z$ is an s.o.p. of A. Then $y_1, \dots, y_{n-1}, x_j z$ is also an s.o.p. of A. So, we are done by Corollary 1 of (2.1). □

Remark. Note that J generates a primary ideal of height 1 in $A/(y_1, \dots, y_{n-1})$. Thus we obtain another proof of Theorem (1.6) as a corollary to the above proposition.

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We now state the two conjectures which will be used in the next two theorems.

1) Canonical Element Conjecture (CEC; Hochster). In elementary terms this conjecture asserts the following ([H2]): Let (A, m, K) be a local ring. Then for every free resolution $F_\bullet$
$$\to A^{s_i} \to A^{s_{i-1}} \to \cdots \to A^{s_0} \to K \to 0$$
of K and for every system of parameters $x_1, \dots, x_n$ of A, if $\phi_\bullet$ is any map of complexes $K_\bullet(\underline{x}; A) \to F_\bullet$ ($K_\bullet(\underline{x}; A)$ denotes the Koszul complex on $x_1, \cdots, x_n$ over A) which lifts the quotient surjection $A/\underline{x} \to K$, then $\phi_n : K_n(\underline{x}; A) \to A^{s_n}$ is non-zero.
2) Direct Summand Conjecture (DSC; Hochster). Let R be a regular local ring and $f : R \hookrightarrow A$ be a module-finite extension. Then f splits as an R-module map ([H1],[H2]).

We already discussed the progress made so far in the study of these conjectures in the introduction. Let R be complete regular local ring of dimension n. Consider a polynomial ring $R[Y_1, \cdots, Y_d]$; let $S = R[Y_1, \cdots, Y_d]/(F_1(Y_1), \cdots, F_t(Y_t))$, where each $F_i(Y)$ is a monic irreducible polynomial of degree d_i in $R[Y]$. Then S is a complete local ring—a complete intersection of dimension n. Moreover S is a free R-module of rank $d_1 d_2 \cdots d_t$. It is not difficult to see that any module-finite extension domain A of R can be obtained as S/P, where P is a minimal prime of S, for some such choice of S as above.

Let x ($\neq 0$) be any zero-divisor in S. Write $I = Ann_S xS$. Then we have the following theorem.

3.1. Theorem. *The direct summand conjecture holds over* R *if and only if* I *contains a minimal generator of* S *over* R*, for every such* x *and* S *as described above.*

Proof. Write $A = S/xS$. Then by Proposition (1.1), the direct summand conjecture holds for A if and only if $I \not\subset mS$, i.e. if and only if I contains a minimal generator of S. On the other hand, for any module finite extension $R \hookrightarrow A$, A a domain, we can construct S as above, and we will have a minimal prime ideal P of S such that $A = S/P$. Choose an $x \in P - \bigcup \mathfrak{q}_{\mathfrak{q} \in Ass(S) - P}$. Then
$$\Omega = \mathrm{Hom}_S(A, S) = \mathrm{Hom}_S(S/xS, S) = I.$$
Hence we are done by Proposition (1.1). □

The above observation inspires us to raise the following question (a bit weaker than the direct summand conjecture!)

Question. Given R, S as above, does S possess a zero-divisor which is also a minimal generator of S over R?

At present, we have the following answer:

3.2. Theorem. *The answer to the above question is in the affirmative when R contains a field or when S contains a minimal prime P such that S/P is normal.*

Proof. The case when R contains a field follows immediately from Theorem 3.1.

Now, let us suppose that S has a minimal prime P such that S/P $(= A)$ is normal. Let $\Omega = \mathrm{Hom}_S(A, S)$. Then S/Ω satisfies CEC. (For a proof, we refer the reader to Theorem 2.5 of [D4].) Hence S/Ω satisfies the direct summand conjecture. Note that over any local ring A, CEC $\Longrightarrow$ DSC. Since $P = \mathrm{Hom}_S(S/\Omega, S)$, by Proposition (1.1), P is not contained in mS, i.e., P contains a minimal generator of S over R. Hence the result. □

Remark. The conclusion of Theorem 3.2 fails in general if R is not assumed to be a regular local ring. One can construct counterexamples even over normal hypersurfaces. This makes one wonder whether the above question is equivalent to DSC. At present, I don't know the answer.

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Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, Illinois 61801,

E-mail address: dutta@math.uiuc.edu